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The quadratic-form identity for constructing the Hamiltonian structure of integrable systems

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Abstract

A usual loop algebra, not necessarily the matrix form of the loop algebra $\tilde{A}_{n-1}$, is also made use of for constructing linear isospectral problems, whose compatibility conditions exhibit a zero-curvature equation from which integrable systems are derived. In order to look for the Hamiltonian structure of such integrable systems, a quadratic-form identity is created in the present paper whose special case is just the trace identity; that is, when taking the loop algebra $\tilde{A}_1$, the quadratic-form identity presented in this paper is completely consistent with the trace identity.

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1. Introduction

The existing linear isospectral problems are mostly expressed by the loop algebra $\tilde{A}_1$, i.e.

$$\begin{cases} \psi_x = U\psi, \quad U = U(\lambda, u), \qquad V = V(\lambda, u) \in \tilde{A}_1, \\ \psi_t = V\psi, \quad \psi = (\psi_1, \psi_2)^T, \quad u = (u_1, u_2, \ldots, u_p)^T, \quad \lambda_t = 0, \end{cases} \tag{1}$$

whose compatibility condition $\psi_{xt} = \psi_{tx}$ exhibits the following zero-curvature equation:

$$U_t - V_x + [U, V] = 0. \tag{2}$$

The nonlinear evolution equation derived from (2)

$$u_t = K(u) \tag{3}$$

is known as Lax integrable. In order to write equation (3) as a Hamiltonian form, the famous trace identity was established in [1–3] as follows:

$$\frac{\delta}{\delta u_i}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\left(\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle\right), \qquad i = 1, 2, \ldots, p, \quad \gamma = \text{const}. \tag{4}$$

This simple and efficient approach has been applied to a host of continuous and discrete integrable systems for constructing Hamiltonian structure which shows that the approach is a powerful tool and holds for arbitrary $\tilde{A}_{n-1}$.

The starting point of the above approach is to choose the functional for two elements A and B of $\tilde{A}_1$

$$f(A, B) = \langle A, B\rangle = \mathrm{tr}(AB) \tag{5}$$

which possesses the following features: symmetry:

$$\langle A, B\rangle = \langle B, A\rangle, \tag{6}$$

bilinear relation:

$$\langle \alpha_1 A_1 + \alpha_2 A_2, B\rangle = \alpha_1\langle A_1, B\rangle + \alpha_2\langle A_2, B\rangle, \tag{7}$$

simplicity of the variational calculation:

$$\nabla_B\langle A, B\rangle = A, \qquad \nabla_B\langle A, B_x\rangle = -A_x, \tag{8}$$

communication:

$$\langle [A, B], C\rangle = \langle A, [B, C]\rangle, \qquad A, B, C \in \tilde{A}_1. \tag{9}$$

Again a proper functional is designed and its variational calculation is given a constrained condition so that the trace identity (4) is obtained. Obviously, if U and V in the linear isospectral problems are not of matrix form, the trace identity is invalid for establishing the Hamiltonian structure. It is our primary purpose to overcome the limitations.

2. A general isospectral problem

Let G be an s-dimensional Lie algebra with the basis

$$e_1, e_2, \ldots, e_s, \tag{10}$$

and the corresponding loop algebra $\tilde{G}$ possesses the basis as follows:

$$e_i(m) = e_i\lambda^m, \qquad i = 1, 2, \ldots, s, \quad m = 0, \pm1, \pm2, \ldots, \quad [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}. \tag{11}$$

Note that

$$\partial = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \tag{12}$$

where α_i are arbitrary constants, $i = 1, 2, \ldots, n$.

For the scalar function a or the element of $\tilde{G}$, denote

$$a_\partial = \partial a = \sum_{i=1}^{n} \alpha_i \frac{\partial a}{\partial x_i}. \tag{13}$$

In terms of $\tilde{G}$, we establish an isospectral problem, but the form (1) cannot be used again,

$$\begin{cases} \psi_\partial = [U, \psi], & U, V, \psi \in \tilde{G}, \\ \psi_t = [V, \psi], & \lambda_t = 0. \end{cases} \tag{14}$$

The compatibility $\psi_{\partial t} = \psi_{t\partial}$ leads to the zero-curvature equation

$$U_t - V_\partial + [U, V] = 0. \tag{15}$$

For λ and u_i $(i = 1, 2, \ldots, p)$ in $U = U(\lambda, u) = \sum_{i=1}^{s} U_i e_i$, we should define the proper rank numbers denoted by rank(λ) and rank(u_i) so that rank$(U_i e_i) = \alpha =$ const, $1 \leqslant i \leqslant s$, and simultaneously we call U the same rank, or homogeneous in rank, denoted by

$$\mathrm{rank}(U) = \mathrm{rank}(\partial) = \mathrm{rank}\left(\frac{\partial}{\partial x_i}\right) = \alpha, \qquad i = 1, 2, \ldots, n. \tag{16}$$

Taking

$$V = \sum_{m \geqslant 0} V_m \lambda^{-m}, \qquad V_m = \sum_{i=1}^{s} V_{mi} e_i \in G, \tag{17}$$

where V_{mi} are scalar functions, and solving the stationary zero-curvature equation

$$V_\partial = [U, V] \tag{18}$$

gives rise to the cycled relations among V_m; it follows from (15) that an integrable hierarchy of evolution equations is worked out.

Assume that the rank(V_m) is given and let rank$(V_m \lambda^{-m})$ be a constant, i.e.,

$$\mathrm{rank}(V_m \lambda^{-m}) = \eta = \mathrm{const}, \qquad m \geqslant 0. \tag{19}$$

V is called the same rank, denoted by

$$\mathrm{rank}(V) = \eta. \tag{20}$$

Let the two arbitrary solutions V and $\bar{V}$ of equation (18) with the same rank have a linear relation

$$\bar{V} = \gamma V, \qquad \gamma = \mathrm{const}. \tag{21}$$

In what follows, relation (21) will be used when deducing the quadratic-form identity.

3. The quadratic-form identity

Let $\tilde{G}$ be the loop algebra in the previous section,

$$a = \sum_{i=1}^{s} a_i e_i, \qquad b = \sum_{i=1}^{s} b_i e_i \in \tilde{G}, \qquad [a, b] = \sum_{i=1}^{s} c_i e_i, \tag{22}$$

representing (22) as the coordinate forms

$$a = (a_1, a_2, \ldots, a_s)^T, \qquad b = (b_1, b_2, \ldots, b_s)^T, \qquad [a, b] = (c_1, c_2, \ldots, c_s)^T, \tag{23}$$

then $\tilde{G}$ can be expressed by

$$\tilde{G} = \left\{ a = (a_1, a_2, \ldots, a_s)^T, a_i = \sum_m a_{im} \lambda^m, 1 \leqslant i \leqslant s \right\} \tag{24}$$

with the commuting operation

$$[a, b] = (c_1, c_2, \ldots, c_s)^T.$$

For $a, b \in \tilde{G}$, define their functional $\{a, b\}$ by

$$\{a, b\} = a^T F b, \tag{25}$$

where $F = (f_{ij})_{s \times s}$ is a symmetric constant matrix, i.e. $F^T = F$.

It is easy to find that $\{a, b\}$ satisfies the symmetry

$$\{a, b\} = \{b, a\}, \tag{26}$$

and the bilinear relation

$$\{\alpha_1 a_1 + \alpha_2 a_2, b\} = \alpha_1\{a_1, b\} + \alpha_2\{a_2, b\}. \tag{27}$$

The gradient $\nabla_b\{a, b\}$ of the functional $\{a, b\}$ is defined by

$$\frac{\partial}{\partial\epsilon}\{a, b + \epsilon V\}|_{\epsilon=0} = \{\nabla_b\{a, b\}, V\}, \tag{28}$$

where $V = (V_1, V_2, \ldots, V_s)^T$.

From (27), it is easy to calculate that

$$\nabla_b\{a, b\} = a. \tag{29}$$

Due to $\partial^* = -\partial$, we have

$$\nabla_b\{a, b_\partial\} = \nabla_b\{-a_\partial, b\} = -a_\partial, \tag{30}$$

i.e. the two operations (8) hold according to the functional $\{a, b\}$. If we take

$$\frac{\partial}{\partial\epsilon}\{a, b + \epsilon V\}|_{\epsilon=0} = (\nabla_b\{a, b\}, V) = \left(\frac{\delta\{a, b\}}{\delta b}, V\right) = \sum_{i=1}^{s} \frac{\delta\{a, b\}}{\delta b_i} V_i, \tag{31}$$

then

$$\nabla_b\{a, b\} = Fa, \qquad \nabla_b\{a, b_\partial\} = -Fa_\partial. \tag{32}$$

Equations (29) and (30) are associated with (25), which we shall use in what follows.

By applying the functional (5), it is easy to deduce the property (9). In terms of $\{a, b\}$, expression (9) can be represented as

$$\{[a, b], c\} = \{a, [b, c]\}. \tag{33}$$

There is an open problem: which symmetric constant matrix $F = (f_{ij})_{s\times s}$ can satisfy expression (33)? Let

$$[a, b]^T = a^T R(b) = -[b, a]^T = -b^T R(a), \tag{34}$$

since $[a, b]^T$ is known, $R(b)$ is a determined $s \times s$ matrix which plays an important role in the paper. Equation (33) can be written as

$$a^T R(b) F c = a^T F (b^T R(c))^T = a^T F(-c^T R(b))^T = a^T(-F R^T(b))c.$$

Due to a and c being arbitrary, we have

$$R(b)F = -F R^T(b) = -(R(b)F)^T \tag{35}$$

which exhibits that $R(b)F$ is an anti-symmetric matrix while F is a symmetric constant matrix.

If (35) had a solution $F \neq 0$, then (33) could hold and the functional $\{a, b\}$ could have the properties (6)–(9).

Set $u = (u_1, u_2, \ldots, u_p)^T$, $u_i = u_i(t, x_1, x_2, \ldots, x_n)$ are the smooth scalar functions, $f(a_1, a_2, \ldots, a_m)$ stands for a functional, $a_k = a_k(u)$ is the functional of u. For the convenience of writing, take $m = 2$, $a = (a_1, a_2)^T$, $f(a_1, a_2) = f(a_1(u), a_2(u)) = f(u)$.

Proposition. *If* $\nabla_a f = \frac{\delta f}{\delta a} = \left(\frac{\delta f}{\delta a_1}, \frac{\delta f}{\delta a_2}\right)^T = 0$, *then*

$$\frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u_1}, \frac{\delta f}{\delta u_2}, \ldots, \frac{\delta f}{\delta u_p}\right)^T = 0. \tag{36}$$

Proof. From the definition of $\nabla_a f$, we have

$$\nabla_a f = \frac{\delta f}{\delta a} = \left(\frac{\delta f}{\delta a_1}, \frac{\delta f}{\delta a_2}\right)^T,$$

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} f(a(u+\epsilon l))|_{\epsilon=0} = \left(\frac{\delta f}{\delta u}, l\right) \\
&= \frac{\partial}{\partial \epsilon}\{f(a_1(u+\epsilon l), a_2(u)) + f(a_1(u), a_2(u+\epsilon l))\}|_{\epsilon=0} \\
&= \sum_{(\beta_1,\beta_2,\ldots,\beta_n)} \left\{ \frac{\partial f}{\partial a_1^{(\beta_1,\beta_2,\ldots,\beta_n)}} \frac{\partial}{\partial \epsilon} a_1^{(\beta_1,\beta_2,\ldots,\beta_n)}(u+\epsilon l)|_{\epsilon=0} \right. \\
&\quad \left. + \frac{\partial f}{\partial a_2^{(\beta_1,\beta_2,\ldots,\beta_n)}} \frac{\partial}{\partial \epsilon} a_2^{(\beta_1,\beta_2,\ldots,\beta_n)}(u+\epsilon l)|_{\epsilon=0} \right\},
\end{aligned}$$

where

$$a_k^{(\beta_1,\beta_2,\ldots,\beta_n)} = \frac{\partial^{\beta_1+\beta_2+\cdots+\beta_n} a_k}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}}, \qquad k = 1, 2, \quad l = (l_1, l_2, \ldots, l_p)^T.$$

Therefore,

$$\begin{aligned}
\left(\frac{\delta f}{\delta u}, l\right) &= \frac{\delta f}{\delta a_1} \frac{\partial}{\partial \epsilon} a_1(u+\epsilon l)|_{\epsilon=0} + \frac{\delta f}{\delta a_2} \frac{\partial}{\partial \epsilon} a_2(u+\epsilon l)|_{\epsilon=0} \\
&= \sum_{(\beta_1,\beta_2,\ldots,\beta_n)} \left\{ \frac{\delta f}{\delta a_1} \left(\frac{\partial a_1}{\partial u^{(\beta_1,\beta_2,\ldots,\beta_n)}}, l^{(\beta_1,\beta_2,\ldots,\beta_n)} \right) + \frac{\delta f}{\delta a_2} \left(\frac{\partial a_2}{\partial u^{(\beta_1,\beta_2,\ldots,\beta_n)}}, l^{(\beta_1,\beta_2,\ldots,\beta_n)} \right) \right\} \\
&= \left(\sum_{(\beta_1,\beta_2,\ldots,\beta_n)} (-1)^{\beta_1+\beta_2+\cdots+\beta_n} \frac{\partial^{\beta_1+\beta_2+\cdots+\beta_n}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}} \right. \\
&\quad \left. \times \left\{ \frac{\delta f}{\delta a_1} \frac{\partial a_1}{\partial u^{(\beta_1,\beta_2,\ldots,\beta_n)}} + \frac{\delta f}{\delta a_2} \frac{\partial a_2}{\partial u^{(\beta_1,\beta_2,\ldots,\beta_n)}} \right\}, l \right),
\end{aligned} \tag{37}$$

where

$$\frac{\partial a_k}{\partial u^{(\beta_1,\beta_2,\ldots,\beta_n)}} = \left(\frac{\partial a_k}{\partial u_1^{(\beta_1,\beta_2,\ldots,\beta_n)}}, \frac{\partial a_k}{\partial u_2^{(\beta_1,\beta_2,\ldots,\beta_n)}}, \ldots, \frac{\partial a_k}{\partial u_p^{(\beta_1,\beta_2,\ldots,\beta_n)}} \right)^T.$$

Hence, if $\frac{\delta f}{\delta a} = 0$, then $\frac{\delta f}{\delta u} = 0$. □

Introduce the functional

$$W = \{V, U_\lambda\} + \{\Lambda, V_\partial - [U, V]\}, \tag{38}$$

where U and V meet (18), while $\Lambda (\in \tilde{G})$ is to be determined. For the variational calculation of W, we give rise to the following constrained conditions:

$$\nabla_V W = U_\lambda - \Lambda_\partial + [U, \Lambda] = 0, \tag{39}$$

$$\nabla_\Lambda W = V_\partial - [U, V] = 0, \tag{40}$$

where U is known, V and Λ are related to U. Moreover, it follows that

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \frac{\delta W}{\delta u_i}, \qquad i = 1, 2, \ldots, p. \tag{41}$$

The u_i of V and U_λ need to be considered in computing the left-hand side of the above formula, while we only consider the u_i of U in computing the right-hand side of the above formula, not necessary to consider the u_i of V and Λ, such that the deducing calculation is derived from the constrained conditions (39), (40) and the proposition (36).

Hence,

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \frac{\delta W}{\delta u_i} = \left\{V, \frac{\partial U_\lambda}{\partial u_i}\right\} + \left\{[\Lambda, V], \frac{\partial U}{\partial u_i}\right\}. \tag{42}$$

From the Jacobi identity and equations (39) and (40), we obtain

$$\begin{aligned}[\Lambda, V]_\partial &= [\Lambda_\partial, V] + [\Lambda, V_\partial] = [U_\lambda + [U, \Lambda], V] + [\Lambda, [U, V]]\\ &= [V, [\Lambda, U]] + [\Lambda, [U, V]] + [U_\lambda, V] = [U, [\Lambda, V]] + [U_\lambda, V].\end{aligned} \tag{43}$$

From (40), we have

$$V_{\lambda\partial} = [U, V_\lambda] + [U_\lambda, V]. \tag{44}$$

Thus, $[\Lambda, V] - V_\lambda = Z$ satisfies

$$Z_\partial = [U, Z].$$

By making use of (21) and $\operatorname{rank}(Z) = \operatorname{rank}(V_\lambda) = \operatorname{rank}\left(\frac{1}{\lambda}V\right)$, due to $\frac{1}{\lambda}V$ being a solution of (40), there exists a constant γ which satisfies

$$[\Lambda, V] - V_\lambda = Z = \frac{\gamma}{\lambda}V. \tag{45}$$

Therefore, (42) can be expressed as

$$\begin{aligned}\frac{\delta}{\delta u_i}\{V, U_\lambda\} &= \left\{V, \frac{\partial U_\lambda}{\partial u_i}\right\} + \left\{V_\lambda, \frac{\partial U}{\partial u_i}\right\} + \frac{\gamma}{\lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\}\\ &= \frac{\partial}{\partial\lambda}\left\{V, \frac{\partial U}{\partial u_i}\right\} + \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\left\{V, \frac{\partial U}{\partial u_i}\right\}\\ &= \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \qquad i = 1, 2, \ldots, p.\end{aligned} \tag{46}$$

We express the above result as follows.

Theorem 1 (The quadratic-form identity). *Assume conditions (16) and (21) hold,* $[a, b]^T = a^T R(b)$*, the symmetric constant matrix* F *turns* $R(b)F$ *into the anti-symmetric matrix. As for the quadratic-form functional* $\{a, b\} = a^T F b$*, the following formula holds:*

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \qquad i = 1, 2, \ldots, p, \tag{47}$$

where γ *is a constant to be determined. We call (47) the quadratic-form identity.*

4. A commuting operator

Set $V_s = \{b = (b_1, b_2, \ldots, b_s)^T\}$ to be an s-dimensional linear space, M_s a set of $s \times s$ matrices. In the course of deducing (47), the operator R from V_s to M_s is introduced by

$$R(b) \in M_s, \qquad b \in V_s \tag{48}$$

so that R is applied to

$$[a, b]^T = a^T R(b). \tag{49}$$

The property of the bilinearity of $[a, b]^T$ requires that R is a linear operator; the anti-symmetry of $[a, b]^T$ requires

$$a^T R(b) = -b^T R(a). \tag{50}$$

For the relation $\{[a, b], c\} = \{a, [b, c]\}$ to hold, F needs to meet the following:

$$R(b)F = -F R^T(b). \tag{51}$$

We find that the linearity of R and equality (50) cannot turn V_s into a Lie algebra with the commuting operation $[a, b]$.

Definition. *Let R be a linear operator from V_s to M_s, and satisfy*

$$R(R^T(b)a) = [R(a), R(b)] = R(a)R(b) - R(b)R(a), \qquad \forall a, b \in V_s, \tag{52}$$

then R is called a commuting operator on V_s. All the commutators on V_s constitute a set denoted by $K(V_s, M_s)$.

Theorem 2. *V_s is a Lie algebra with the commuting operation $[a, b]$ if and only if there exists $R \in K(V_s, M_s)$ so that*

$$[a, b]^T = a^T R(b). \tag{53}$$

Proof. Set $[a, b]$ to be a commutator of the Lie algebra V_s, $[a, b]^T = a^T R(b)$. According to the linearity of $[a, b]$ with respect to b, R is a linear operator from V_s to M_s. Since $[a, b]^T = -[b, a]^T$ we have $a^T R(b) = -b^T R(a)$. By making use of the Jacobi identity, we have

$$\begin{aligned}
&[[a, b], c]^T + [[b, c], a]^T + [[c, a], b]^T \\
&\qquad = a^T R(b)R(c) + b^T R(c)R(a) + b^T R([a, c])) \\
&\qquad = b^T (R(c)R(a) - R(a)R(c) + R([a, c]) \\
&\qquad = 0, \qquad \forall a, b, c \in V_s.
\end{aligned}$$

Since b is arbitrary, the following relation holds:

$$R([a, c]) = R(a)R(c) - R(c)R(a) = [R(a), R(c)], \qquad \forall a, c \in V_s.$$

Since $R([a, c]) = R(R^T(c)a)$, the equality (52) holds.

Conversely, for $\forall R \in K(V_s, M_s)$, we regard

$$[a, b] = R^T(b)a \; ([a, b]^T = a^T R(b)) \tag{54}$$

as a commuting operation of V_s.

Since R is a linear operator, $[a, b]$ is bilinear. In terms of (52), we obtain

$$R(R^T(b)a) = -R(R^T(a)b). \tag{55}$$

Therefore, $R^T(b)a = -R^T(a)b$, that is, $[a, b]$ is anti-symmetric. Condition (52) guarantees that the commuting operation defined by (54) satisfies the Jacobi identity. Hence, V_s is a Lie algebra. □

Corollary. *Let*

$$b = (b_1, b_2, b_3)^T, \qquad R(b) = \begin{pmatrix} \alpha_1 b_2 + \alpha_2 b_3 & \beta_1 b_2 + \beta_2 b_3 & \gamma_1 b_2 + \gamma_2 b_3 \\ -\alpha_1 b_1 + \alpha_3 b_3 & -\beta_1 b_1 + \beta_3 b_3 & -\gamma_1 b_1 + \gamma_3 b_3 \\ -\alpha_2 b_1 - \alpha_3 b_2 & -\beta_2 b_1 - \beta_3 b_2 & -\gamma_2 b_1 - \gamma_3 b_2 \end{pmatrix}, \tag{56}$$

where $\alpha_i, \beta_j, \gamma_k$ are constants to be determined. Then $R \in K(V_3, M_3)$ if and only if

$$\begin{cases} \alpha_2\gamma_1 - \alpha_1\gamma_2 = \beta_1\gamma_3 - \beta_3\gamma_1, \\ \alpha_2\beta_1 - \alpha_1\beta_2 = \beta_3\gamma_2 - \beta_2\gamma_3, \\ \alpha_3\beta_1 - \alpha_1\beta_3 = \alpha_2\gamma_3 - \alpha_3\gamma_2. \end{cases} \tag{57}$$

Given $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$, set $\alpha_2 + \beta_3 \neq 0, \alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$, then there exist the unique solutions $\gamma_1, \gamma_2, \gamma_3$ of the linear equations (57).

Example 1. $V_3 = \tilde{A}_1 = \left\{A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{smallmatrix}\right) = a_1 h + a_2 e + a_3 f = (a_1, a_2, a_3)^T, a_k = \sum_m a_{km}\lambda^m, m = 0, \pm 1, \pm 2, \ldots\right\}$,

$$[A, B]^T = (AB - BA)^T = (a_2b_3 - a_3b_2, 2a_1b_2 - 2a_2b_1, 2a_3b_1 - 2a_1b_3)$$

$$= (a_1, a_2, a_3)\begin{pmatrix} 0 & 2b_2 & -2b_3 \\ b_3 & -2b_1 & 0 \\ -b_2 & 0 & 2b_1 \end{pmatrix} = a^T R(b), \tag{58}$$

where $\alpha_1 = \alpha_2 = 0, \alpha_3 = 1, \beta_1 = 2, \beta_2 = \beta_3 = 0, \gamma_2 = -2, \gamma_1 = \gamma_3 = 0$, $R(b)$ meets (57). It is easy to verify that

$$F = c\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (\text{taking } c = 1) \tag{59}$$

satisfies $(R(b)F)^T = -R(b)F$.

$$\begin{aligned} &a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \leftrightarrow A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \leftrightarrow B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}, \\ &\langle A, B\rangle = \operatorname{tr}(AB) = 2a_1b_1 + a_2b_3 + a_3b_2, \qquad \{a, b\} = a^T F b = 2a_1b_1 + a_3b_2 + a_2b_3, \\ &\langle A, B\rangle = \{a, b\}. \end{aligned} \tag{60}$$

The equality (60) shows that (47) and (4) are completely consistent when taking $V_3 = \tilde{A}_1$.

Example 2. $V_3 = R_3 = \{a = a_1 i + a_2 j + a_3 k = (a_1, a_2, a_3)^T\}$, the vector product of a and b in R_3 is as follows:

$$a \times b = \det\begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

$$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^T.$$

Obviously, we have

$$(a \times b)^T = (a_1, a_2, a_3)\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}, \tag{61}$$

where

$$R(b) = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} \tag{62}$$

meets (57), and R_3 is a Lie algebra along with the commuting operation $[a, b] = a \times b$. Since $R^T(b) = -R(b)$, we may take $F = \operatorname{diag}(1, 1, 1)$ such that

$$\{a, b\} = a_1b_1 + a_2b_2 + a_3b_3. \tag{63}$$

Example 3. The loop algebra $V_6 = \{a = (a_1, a_2, \ldots, a_6)^T, a_k = \sum_m a_{km}\lambda^m\}$,

$$[a, b]^T = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1, a_2b_6 - a_6b_2 + a_5b_3 - a_3b_5,\\ a_3b_4 - a_4b_3 + a_6b_1 - a_1b_6, a_1b_5 - a_5b_1 + a_4b_2 - a_2b_4), \tag{64}$$

$$R(b) = \begin{pmatrix} 0 & -b_3 & b_2 & 0 & -b_6 & b_5 \\ b_3 & 0 & -b_1 & b_6 & 0 & -b_4 \\ -b_2 & b_1 & 0 & -b_5 & b_4 & 0 \\ 0 & 0 & 0 & 0 & -b_3 & b_2 \\ 0 & 0 & 0 & b_3 & 0 & -b_1 \\ 0 & 0 & 0 & -b_2 & b_1 & 0 \end{pmatrix}, \tag{65}$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{66}$$

$$\{a, b\} = a^T F b = a_1b_1 + a_1b_4 + a_4b_1 + a_2b_2 + a_2b_5 + a_5b_2 + a_3b_6 + a_6b_3 + a_3b_3. \tag{67}$$

The two subalgebras of V_6 read $G_1 = \{a = (a_1, a_2, a_3, 0, 0, 0)^T\}$, $G_2 = \{b = (0, 0, 0, b_1, b_2, b_3)^T\}$, which satisfy

$$V_6 = G_1 \dotplus G_2, [G_1, G_2] \subset G_2. \tag{68}$$

Integrable couplings are a quite new topic in soliton theory [4, 5]. By employing (68), we may derive integrable couplings of the known integrable hierarchies of soliton equations with the help of V_6 [6–8].

5. Applications of the quadratic-form identity

We make use of the loop algebra V_6 and the resulting functional (67) to derive a hierarchy.

Let $U = (\lambda, u_1, u_2, 0, u_3, u_4)^T$, $\operatorname{rank}(\lambda) = \operatorname{rank}(u_k) = \operatorname{rank}(U) = \operatorname{rank}(\partial) = 1$, $1 \leqslant k \leqslant 4$. Taking $V = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)}, b^{(6)})^T$, $b^{(k)} = \sum_{m\geqslant 0} b_m^{(k)}\lambda^{-m}$, $k = 1, 2, \ldots, 6$, and solving the equation

$$V_\partial = [U, V] \tag{69}$$

gives

$$\begin{cases} b_{m\partial}^{(1)} = u_1 b_m^{(3)} - u_2 b_m^{(2)}, \quad b_{m\partial}^{(2)} = u_2 b_m^{(1)} - b_{m+1}^{(3)}, \\ b_{m\partial}^{(3)} = b_{m+1}^{(2)} - u_1 b_m^{(1)}, \quad b_{m\partial}^{(4)} = u_1 b_m^{(6)} - u_4 b_m^{(2)} + u_3 b_m^{(3)} - u_2 b_m^{(5)}, \\ b_{m\partial}^{(5)} = -b_{m+1}^{(6)} + u_2 b_m^{(4)} + u_4 b_m^{(1)}, \\ b_{m\partial}^{(6)} = b_{m+1}^{(5)} - u_3 b_m^{(1)} - u_1 b_m^{(4)}, \\ b^{(1)} = \beta = \text{const}, \quad b_0^{(k)} = 0, \quad 2 \leqslant k \leqslant 6, \quad b_1^{(2)} = \beta u_1, \quad b_1^{(3)} = \beta u_2, \quad b_1^{(5)} = \beta u_3, \\ b_1^{(6)} = \beta u_4, \quad b_1^{(1)} = b_1^{(4)} = 0, \quad b_2^{(1)} = -\frac{\beta}{2}\left(u_1^2 + u_2^2\right), \quad b_2^{(2)} = \beta u_{2\partial}, \\ b_2^{(3)} = -\beta u_{1\partial}, \quad b_2^{(4)} = -\beta(u_1u_3 + u_2u_4), \quad b_2^{(5)} = \beta u_{4\partial}, \\ b_2^{(6)} = -\beta u_{3\partial}, \quad \operatorname{rank}\left(b_m^{(k)}\right) = m, \quad \operatorname{rank}(V) = 0. \end{cases} \tag{70}$$

Noting $V_+^{(n)} = \sum_{m=0}^{n} \left(b_m^{(1)}, b_m^{(2)}, b_m^{(3)}, b_m^{(4)}, b_m^{(5)}, b_6^{(6)}\right)^T \lambda^{n-m}$, $V_-^{(n)} = \lambda^n V - V_+^{(n)}$, equation (69) can be written as

$$-V_{+\partial}^{(n)} + \left[U, V_+^{(n)}\right] = V_{-\partial}^{(n)} - \left[U, V_-^{(n)}\right].$$

The terms on the left-hand side in the above equality are of degree $\geqslant 0$, while the terms on the right-hand side are of degree $\leqslant 0$. Therefore,

$$-V_{+\partial}^{(n)} + \left[U, V_+^{(n)}\right] = -\left(0, -b_{n+1}^{(3)}, b_{n+1}^{(2)}, 0, -b_{n+1}^{(6)}, b_{n+1}^{(5)}\right)^T.$$

Denoting $V^{(n)} = V_+^{(n)}$, the following zero-curvature equation

$$U_t - V_\partial^{(n)} + [U, V^{(n)}] = 0$$

gives the integrable system

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} -b_{n+1}^{(3)} \\ b_{n+1}^{(2)} \\ -b_{n+1}^{(6)} \\ b_{n+1}^{(5)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1}^{(2)} + b_{n+1}^{(5)} \\ b_{n+1}^{(3)} + b_{n+1}^{(6)} \\ b_{n+1}^{(2)} \\ b_{n+1}^{(3)} \end{pmatrix} = J P_{n+1}. \tag{71}$$

According to the quadratic-form identity, we obtain

$$\frac{\delta}{\delta u}(b^{(1)} + b^{(4)}) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \left(\lambda^\gamma \begin{pmatrix} b^{(2)} + b^{(5)} \\ b^{(3)} + b^{(6)} \\ b^{(2)} \\ b^{(3)} \end{pmatrix} \right). \tag{72}$$

Comparison of the coefficients of λ^{-n-2} yields

$$\frac{\delta}{\delta u}\left(b_{n+2}^{(1)} + b_{n+2}^{(4)}\right) = (\gamma - n - 1) \begin{pmatrix} b_{n+1}^{(2)} + b_{n+1}^{(5)} \\ b_{n+1}^{(3)} + b_{n+1}^{(6)} \\ b_{n+1}^{(2)} \\ b_{n+1}^{(3)} \end{pmatrix}. \tag{73}$$

Taking $n = 0$ leads to $\gamma = 0$. Therefore,

$$P_{n+1} = \frac{\delta H_{n+1}}{\delta u}, \qquad H_{n+1} = -\frac{b_{n+2}^{(1)} + b_{n+2}^{(4)}}{n+1}, \qquad n \geqslant 0. \tag{74}$$

The system (71) can be written as

$$u_t = J P_{n+1} = J \frac{\delta H_{n+1}}{\delta u}, \qquad n \geqslant 0. \tag{75}$$

From (70), an operator L meets

$$P_{n+1} = L P_n, \tag{76}$$

where

$$L = \begin{pmatrix} -u_1 \partial^{-1} u_2 & \partial + u_1 \partial^{-1} u_1 & -u_3 \partial^{-1} u_2 - u_1 \partial^{-1} u_4 & u_3 \partial^{-1} u_1 + u_1 \partial^{-1} u_3 \\ -\partial - u_2 \partial^{-1} u_2 & u_2 \partial^{-1} u_1 & -u_4 \partial^{-1} u_2 - u_2 \partial^{-1} u_4 & u_4 \partial^{-1} u_1 + u_2 \partial^{-1} u_3 \\ 0 & 0 & -u_1 \partial^{-1} u_2 & \partial + u_1 \partial^{-1} u_1 \\ 0 & 0 & -\partial - u_2 \partial^{-1} u_2 & u_2 \partial^{-1} u_1 \end{pmatrix}.$$

Hence, (75) can be written again as

$$u_t = JL^n \begin{pmatrix} \beta(u_1+u_3) \\ \beta(u_2+u_4) \\ \beta u_1 \\ \beta u_2 \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \qquad n \geqslant 0. \tag{77}$$

We observe that

$$JL = L^* J. \tag{78}$$

From (70), the scalars $b^{(1)}$, $b^{(2)}$ and $b^{(3)}$ are independent of u_3 and u_4; that is to say, the first and second equations in (71) have no relations with u_3 and u_4. Therefore, we obtain

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t &= \begin{pmatrix} -b^{(3)}_{n+1} \\ b^{(2)}_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b^{(2)}_{n+1} \\ b^{(3)}_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 \\ -\partial - u_2\partial^{-1}u_2 & u_2\partial^{-1}u_1 \end{pmatrix}^n \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix}, \qquad n \geqslant 0. \end{aligned} \tag{79}$$

Assuming $u_3 = u_4 = 0$, we may take $b^{(4)} = b^{(5)} = b^{(6)} = 0$; then the system (77) casts into the reduced system (79), which is an integrable hierarchy of equations. The later two equations in (77) concretely contain u_1 and u_2; therefore, it is a type of integrable coupling of the system (79). We observe that the system (79) is similar to the AKNS hierarchy; however, it is a far cry from the AKNS hierarchy. Hence, we call the system (79) a modified-AKNS hierarchy, which is denoted by m-AKNS hierarchy. It is easy to obtain from (79) that

$$\begin{pmatrix} b^{(2)}_{n+1} \\ b^{(3)}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\delta}{\delta u_1} \\ \frac{\delta}{\delta u_2} \end{pmatrix} \left(-\frac{b^{(1)}_{n+2}}{n+1} \right). \tag{80}$$

Since the equality (78) holds, the Hamiltonian functions H_l ($l \geqslant 1$) in (77) are involutive each other and each H_l is the common conserved density of (77). Therefore, the hierarchy (77) is a Liouville integrable hierarchy.

The m-AKNS–KN hierarchy was obtained in [7] whose Hamiltonian structure could not be worked out because the trace identity was not suitable. Now we are satisfied that the problem is overcome by using the quadratic-form identity presented in this paper. We recall that the integrable couplings of the AKNS hierarchy and the KN hierarchy were obtained by using the subalgebras of the loop algebra $\tilde{A}_2$ in [5, 6]. It seems we can use the trace identity to get the Hamiltonian structure of the above integrable coupling, but some equalities just as $0 = 0$ were presented by making use of the trace identity, which shows that the trace identity holds but is invalid.

The system (77) is the integrable coupling of the m-AKNS hierarchy (79) and possesses the Hamiltonian structure. Therefore, it is the first example with two such features.

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